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A flux theorem.

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## A FLUX THEOREM

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**Summary**

In the literature <sup>1)</sup> <sup>2)</sup> some formulae are known which give the total flux incident on an area originating from a surface source. These formulae are derived again by aid of a flux theorem which transforms the original double surface integral into a double line integral over the boundaries of both areas.

In illuminating engineering and elsewhere, one of the many problems is the calculation of the total flux incident on an area and originating from a surface source. We assume a perfectly diffusing source, i.e. the emitted flux obeys Lambert's law.

Let  $d\mathbf{A}_1$  be the oriented surface element of the source and  $d\mathbf{A}_2$  the oriented surface element of the illuminated area (fig. 1), then, if  $\mathbf{r}$  is the radius vector connecting both elements and if  $L$  is the luminosity of the source, the total flux incident on the illuminated area is

$$E = \frac{L}{\pi} \iint r^{-4} \mathbf{r} \cdot d\mathbf{A}_1 \mathbf{r} \cdot d\mathbf{A}_2. \quad (1)$$

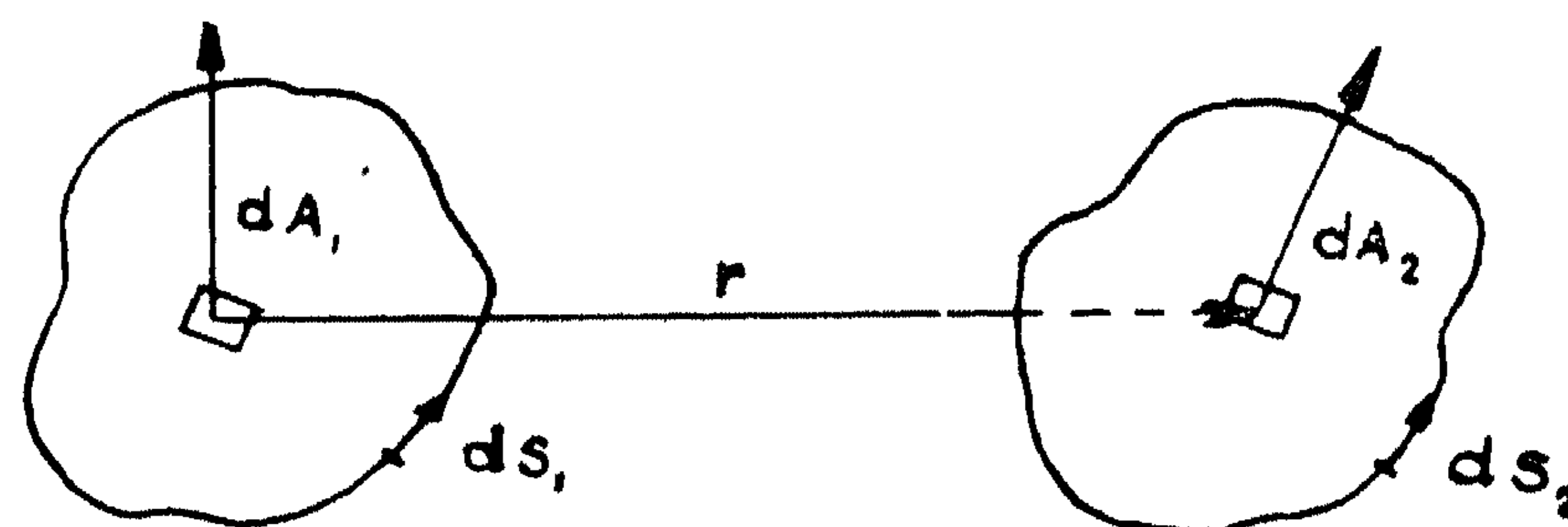


Fig. 1

This double surface integral can be transformed into the double line integral

$$E = -\frac{L}{2\pi} \oint \oint \log r \, ds_1 \cdot ds_2, \quad (2)$$

where  $ds_1$  and  $ds_2$  are the oriented line elements of the boundaries of the surface source and the illuminated area. Since  $\oint \mathbf{a} \cdot d\mathbf{s} = 0$ , we have  $\oint \oint ds_1 \cdot ds_2 = 0$ , and hence  $\log r$  in (2) may be replaced by

$\log(r/r_0)$  where  $r_0$  is an arbitrary constant. This means that  $r$  in  $\log r$  can be regarded as dimensionless.

In the proof we use Stokes' theorem

$$\int \nabla \times \mathbf{v} \cdot d\mathbf{A} = \oint \mathbf{v} \cdot d\mathbf{s}.$$

Application of the theorem to the integration with respect to  $A_2$  yields

$$\begin{aligned} \oint \oint \log r \, d\mathbf{s}_1 \cdot d\mathbf{s}_2 &= \oint \int (\nabla \times \log r \, d\mathbf{s}_1) \cdot d\mathbf{A}_2 \\ &= \oint \int (r^{-2} \mathbf{r} \times d\mathbf{s}_1) \cdot d\mathbf{A}_2. \end{aligned}$$

In this integration the elements  $d\mathbf{s}_2$  and  $d\mathbf{A}_2$  have been „Aufpunkt“. If we apply Stokes' theorem once again with respect to  $A_1$ , the elements  $d\mathbf{s}_1$  and  $d\mathbf{A}_1$  become „Aufpunkt“. This means that we have to invert the direction of  $\mathbf{r}$ . Hence, using the vector formulae

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b}, \quad \text{and}$$

$$\begin{aligned} \nabla \times (r^{-2} \mathbf{r} \times \mathbf{a}) &= \mathbf{a} \cdot \nabla r^{-2} \mathbf{r} - \mathbf{a} \nabla \cdot r^{-2} \mathbf{r} \\ &= r^{-2} \mathbf{a} - 2r^{-4} \mathbf{r} \cdot \mathbf{a} \mathbf{r} - \mathbf{a} r^{-2} = -2r^{-4} \mathbf{r} \cdot \mathbf{a} \mathbf{r}, \end{aligned}$$

we find

$$\begin{aligned} \oint \oint \log r \, d\mathbf{s}_1 \cdot d\mathbf{s}_2 &= \int \oint (r^{-2} \mathbf{r} \times d\mathbf{A}_2) \cdot d\mathbf{s}_1 \\ &= \int \int [\nabla \times (r^{-2} \mathbf{r} \times d\mathbf{A}_2)] \cdot d\mathbf{A}_1 \\ &= -2 \int \int r^{-4} \mathbf{r} \cdot d\mathbf{A}_2 \mathbf{r} \cdot d\mathbf{A}_1. \end{aligned}$$

It is an important consequence of (2) that the actual shapes of the surfaces  $A_1$  and  $A_2$  are irrelevant, the flux  $E$  depending only on their boundaries.

Eq. (2) is very suitable for the calculation of the total flux in special cases.

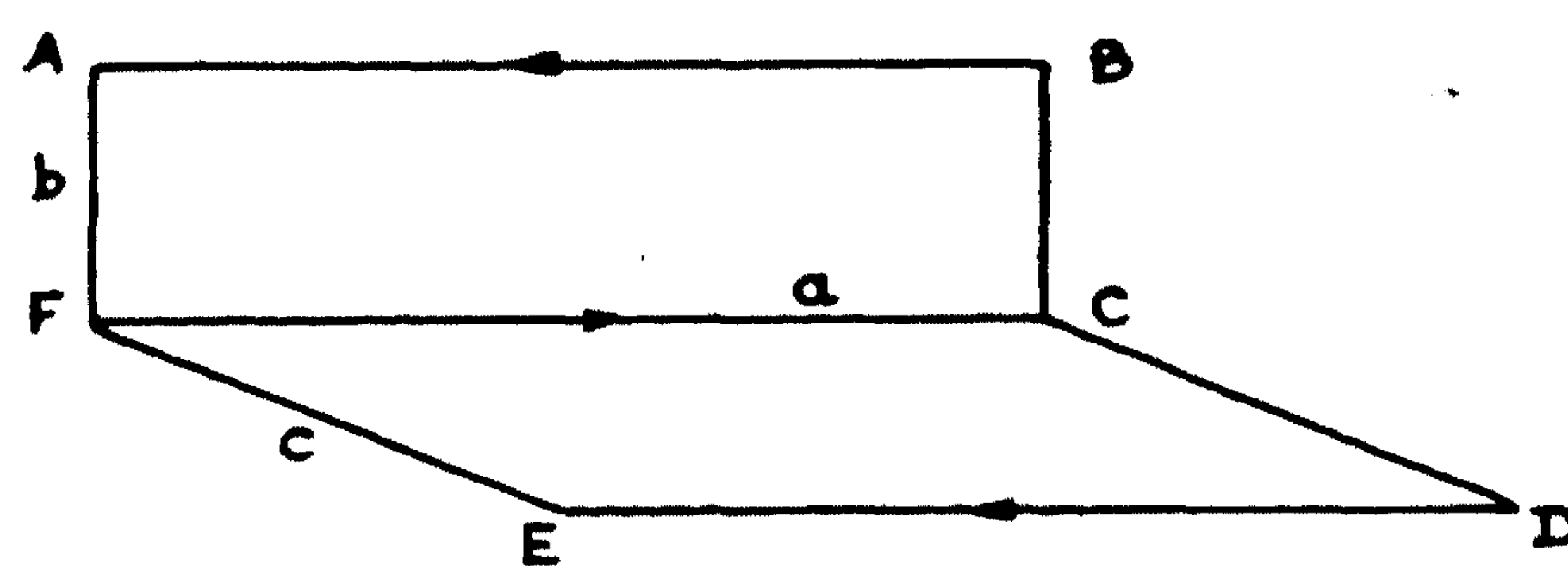


Fig. 2

a) Let ABCF be the surface source and let CDEF be the illuminated area which is perpendicular to the source (fig. 2). Then

$$-d\mathbf{s}_1 \cdot d\mathbf{s}_2 = ds_{AB} ds_{FC} + ds_{DE} ds_{FC} - ds_{AB} ds_{FC} - ds_{FC} ds_{FC}.$$

If we now define

$$\varphi(p) = \frac{1}{2} \int_0^a \int_0^a \log r \, ds_1 ds_2,$$



where  $ds_1$  and  $ds_2$  are line elements of two parallel lines which are a distance  $p$  apart, we find

$$E = \frac{L}{\pi} [\varphi(b) + \varphi(c) - \varphi(\sqrt{b^2 + c^2}) - \varphi(0)],$$

or, with  $a^2\phi(p) = \varphi(p) - \varphi(0)$

$$E = \frac{a^2L}{\pi} [\phi(b) + \phi(c) - \phi(\sqrt{b^2 + c^2})].$$

For the function  $\phi(p)$  we find

$$\varphi(p) = \frac{1}{4}[(a^2 - p^2)\log(a^2 + p^2) - a^2 + p^2\log p^2] + ap \arctan \frac{a}{p},$$

$$\varphi(0) = \frac{1}{4}[a^2\log a^2 - a^2],$$

$$\phi(p) = \frac{1}{2}\left[\left(\frac{p}{a}\right)^2 \log \frac{p}{\sqrt{a^2 + p^2}} - \log \frac{a}{\sqrt{a^2 + p^2}}\right] + \frac{p}{a} \arctan \frac{a}{p},$$

or, putting  $a = p \tan \omega$ ,

$$\phi(p) = \frac{1}{2}(\cot^2 \omega \log \cos \omega - \log \sin \omega) + \omega \cot \omega$$

in accordance with Moon<sup>1</sup>).

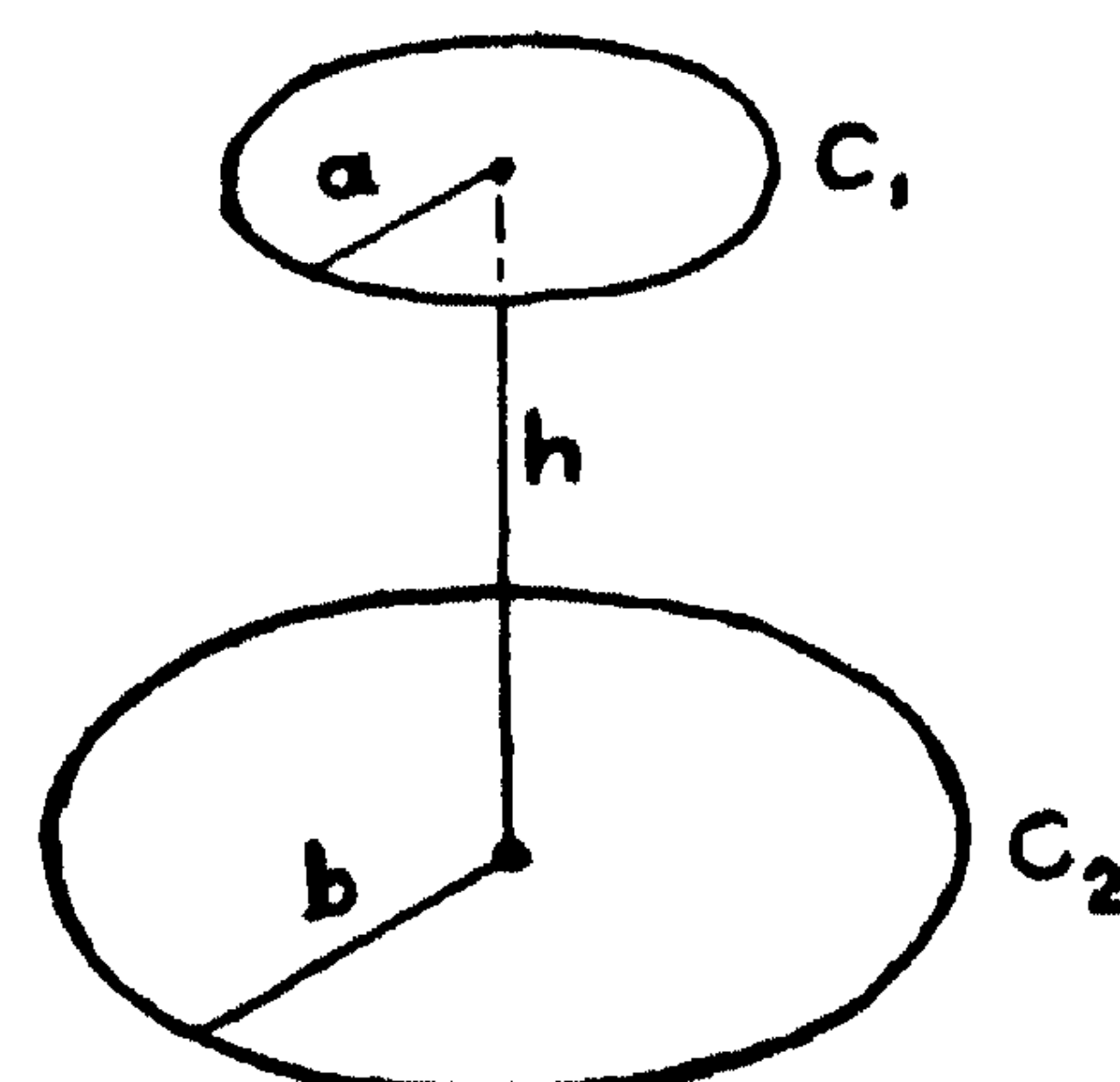


Fig. 3

b) Let  $C_1$  and  $C_2$  be circles in parallel planes (fig. 3) with radii  $a$  and  $b$  and with centres on a common perpendicular. Then we have, if the distance of the planes is  $h$ ,

$$E = -\frac{L}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \log[h^2 + a^2 + b^2 - 2ab \cos \varphi] d\varphi d\psi$$

$$= \frac{h^2L\pi}{4} \left[ \sqrt{1 + \left(\frac{a+b}{h}\right)^2} - \sqrt{1 + \left(\frac{a-b}{h}\right)^2} \right]^2,$$

which differs by a factor  $\pi$  with the result given by Moon and is in agreement with Milne's result<sup>2</sup>).

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#### REFERENCES

- 1) Moon, P., The scientific basis of illuminating engineering, McGraw Hill, section 10.11.
- 2) Milne, E. A., Phil. Mag. **7** (1928) 273.